

# DIOPHANTINE APPROXIMATION, KHINTCHINE'S THEOREM, TORUS GEOMETRY AND HAUSDORFF DIMENSION

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**ABSTRACT.** A general form of the Borel-Cantelli Lemma and its connection with the proof of Khintchine's Theorem on Diophantine approximation and the more general Khintchine-Groshev theorem are discussed. The torus geometry in the planar case allows a relatively direct proof of the planar Groshev theorem for the set of  $\psi$ -approximable points in the plane. The construction and use of Hausdorff measure and dimension are explained and the notion of ubiquity, which is effective in estimating the lower bound of the Hausdorff dimension for quite general lim sup sets, is described. An application is made to obtain the Hausdorff dimension of the set of  $\psi$ -approximable points in the plane when  $\psi(q) = q^{-v}$ ,  $v > 0$ , corresponding to the planar Jarník-Besicovitch theorem.

## 1. DIOPHANTINE APPROXIMATION

Diophantine approximation is a quantitative analysis of the density of the rationals in the reals. It is easy to see from the distribution of the integers  $\mathbb{Z}$  in the real line  $\mathbb{R}$ , that given any  $\alpha \in \mathbb{R}$  and any  $q \in \mathbb{N}$ , there exists a  $p = p(\alpha, q) \in \mathbb{Z}$  such that

$$|q\alpha - p| \leq 1/2 \text{ or } |\alpha - p/q| \leq \frac{1}{2q}.$$

It is possible to do better using continued fractions (see [6, 18]) and thanks to Dirichlet's box argument [19], to obtain a best possible result.

**Theorem 1** (Dirichlet). *Given  $\alpha \in \mathbb{R}$ ,  $N \in \mathbb{N}$ , there exist integers  $p, q$  with  $1 \leq q \leq N$  such that*

$$|\alpha - p/q| \leq \frac{1}{q(N+1)}.$$

Given any  $\xi \in \mathbb{R}$ , the convenient notation

$$\|\xi\| := \min\{|\xi - p| : p \in \mathbb{Z}\}$$

will be used. Thus Dirichlet's theorem implies that given any  $\alpha \in \mathbb{R}$ , there are infinitely many  $q \in \mathbb{N}$  such that

$$\|q\alpha\| = \min\{|q\alpha - p| : p \in \mathbb{Z}\} < \frac{1}{q}.$$

More generally, an error term or *approximation* function  $\psi : \mathbb{N} \rightarrow (0, \infty)$ , where  $\lim_{q \rightarrow \infty} \psi(q) = 0$ , is introduced and the solubility of

$$\|q\alpha\| < \psi(q) \tag{1}$$

considered ( $\psi(q) = 1/q$  in Dirichlet's theorem). Note that although restricting the approximation to rationals  $p/q$  with  $(p, q) = 1$  is natural and indeed is associated with the Duffin-Schaeffer conjecture (see [13]), coprimality does not arise in the present formulation.

The point  $\alpha$  is said to be  *$\psi$ -approximable* if (1) holds for infinitely many  $q \in \mathbb{N}$ . The set  $W(\psi)$  of  $\psi$ -approximable numbers is invariant under translation by integers and so there is no loss of generality in restricting attention to the unit interval and considering

$$\begin{aligned} W(\psi) &:= \{\alpha \in [0, 1] : \|q\alpha\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\} \\ &= \{\alpha \in [0, 1] : |\alpha - p/q| < \psi(q)/q \text{ for infinitely many } p \in \mathbb{Z}, q \in \mathbb{N}\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=0}^q \left( \frac{p}{q} - \frac{\psi(q)}{q}, \frac{p}{q} + \frac{\psi(q)}{q} \right) \cap [0, 1]. \end{aligned}$$

The set  $W(\psi)$  is a lim sup set, as can be seen by letting for each  $q \in \mathbb{N}$  and  $\rho > 0$ ,

$$B(q; \rho) = \bigcup_{p=0}^q \left( \frac{p}{q} - \frac{\rho}{q}, \frac{p}{q} + \frac{\rho}{q} \right) \cap [0, 1],$$

so that  $B(q; \rho)$  is a  $\rho$ -neighbourhood of the resonant set (so called from the connection with the physical phenomenon of resonance)

$$R_q := \left\{ 0, \frac{1}{q}, \dots, \frac{p}{q}, \dots, \frac{q-1}{q}, 1 \right\}$$

and

$$W(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{q=1}^N B(q; \psi(q)) = \limsup_{q \rightarrow \infty} B(q; \psi(q)). \quad (2)$$

Moreover,  $|B(q; \rho)| = 2\rho$ , since the set of points in  $[0, 1]$  satisfying

$$\left| \alpha - \frac{p}{q} \right| < \frac{\rho}{q},$$

shown in Figure 1, has length

$$\frac{\rho}{q} + (q-1) \frac{2\rho}{q} + \frac{\rho}{q} = 2\rho.$$

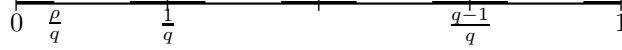


Figure 1. Points  $p/q$  and neighbourhood  $B(q; \rho)$  in  $[0, 1]$ .

We are interested in the ‘size’ of  $W(\psi)$ . The size of a measurable set  $E$  will be interpreted as its Lebesgue measure, denoted by  $|E|$ . The question of the measure of  $W(\psi)$  is almost completely answered by

### 1.1. Khintchine’s theorem.

**Theorem 2.** *The Lebesgue measure of  $W(\psi)$  is given by*

$$|W(\psi)| = \begin{cases} 0, & \text{if } \sum_{k=1}^{\infty} \psi(k) < \infty, \\ 1, & \text{if } \sum_{k=1}^{\infty} \psi(k) = \infty \text{ and } \psi(k) \text{ is non-increasing.} \end{cases}$$

The theorem corresponds to our intuition, as when the approximation function  $\psi$  is large, there is a better chance of the inequality being satisfied and vice-versa (for more details, see [6, 27]). Thus the Lebesgue measure of  $W(\psi)$  is 1 when  $\psi(q) = 1/(q \log q)$  and 0 when  $\psi(q) = 1/(q(\log q)^{1+\varepsilon})$  for any positive  $\varepsilon$ . This ‘0-1’ property is a feature of the metrical theory and reflects its links with probability. Indeed the result is reminiscent of the Borel-Cantelli Lemma from probability theory (see [22]). Let  $E_j, j = 1, 2, \dots$  be a sequence of events in a probability space  $(\Omega, P)$ , let

$$\begin{aligned} E &= \{x \in \Omega : x \in E_j \text{ infinitely often} \} \\ &= \bigcup_{N=1}^{\infty} \bigcap_{r=N}^{\infty} E_r = \limsup_{N \rightarrow \infty} E_N. \end{aligned}$$

Then the lim sup  $E$  is the the set of points lying in infinitely many events  $E_j$  and  $P(E)$  is the probability that infinitely many events  $E_j$  occur.

**Lemma 1** (Borel-Cantelli).

$$P(E) = \begin{cases} 0, & \text{if } \sum_{j=1}^{\infty} P(E_j) < \infty, \\ 1, & \text{if } E_j \text{ totally independent and } \sum_{j=1}^{\infty} P(E_j) = \infty. \end{cases}$$

Borel proved the lemma assuming total independence of the events, when for any distinct events  $E_{j_1}, \dots, E_{j_k}$ ,

$$P(E_{j_1} \cap \dots \cap E_{j_k}) = P(E_{j_1}) \dots P(E_{j_k}).$$

Cantelli observed that independence is not needed when the sum of probabilities converges. In the case of divergence, the result holds under total *quasi*-independence, when for some  $K \geq 1$ ,

$$P(E_{j_1} \cap \dots \cap E_{j_k}) \leq K P(E_{j_1}) \dots P(E_{j_k}).$$

If Lebesgue measure is interpreted as probability, then  $|B(q; \psi(q))|$  corresponds to the probability that a point  $\alpha \in [0, 1]$  falls into  $B(q; \psi(q))$  and the proof of Khintchine's theorem in the case of convergence is essentially that in the Borel-Cantelli Lemma. Indeed, the set of points in  $[0, 1]$  satisfying (1) for a given  $q \in \mathbb{N}$  and  $\rho = \psi(q)$ , shown in Figure 1, has length

$$\frac{\psi(q)}{q} + (q-1) \frac{2\psi(q)}{q} + \frac{\psi(q)}{q} = 2\psi(q).$$

Hence for any  $N \in \mathbb{N}$ ,

$$|W(\psi)| \leq 2 \sum_{q=N}^{\infty} \psi(q) \rightarrow 0 \text{ as } N \rightarrow \infty,$$

whence  $|W(\psi)| = 0$ . Sets of Lebesgue measure 0 will be called *null*.

The case of divergence is more difficult. The Borel-Cantelli Lemma assumes total independence to deal with the divergence case and so is useless. However a more general lower bound for  $\limsup$  set of the sets  $E_j$  is available. It suits our purposes to express the result in terms of the Lebesgue measure  $|E_j|$  of the sets  $E_j$ .

**Theorem 3.** *Let  $E_j$ ,  $j = 1, 2, \dots$ , be a sequence of Lebesgue measurable sets in  $\Omega = [0, 1]^n$  and suppose that*

$$\sum_{j=1}^{\infty} |E_j| = \infty. \quad (3)$$

*Then the Lebesgue measure of  $E := \limsup_{N \rightarrow \infty} E_N := \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k$  satisfies*

$$|E| \geq \limsup_{N \rightarrow \infty} \frac{(\sum_{k=1}^N |E_k|)^2}{\sum_{k=1}^N \sum_{l=1}^N |E_k \cap E_l|}. \quad (4)$$

Because of the importance of the result, a proof will be given. This is based on 'mean and variance' arguments (see [28] and also [19, 27]). It appears as an exercise in [7] and there are numerous variants, e.g., [23].

*Proof.* For each  $n = 1, 2, \dots$ , let  $\nu_n: \Omega \rightarrow [0, \infty]$  be the counting function of the number of  $E_j$  into which  $x$  falls, so that

$$\nu_n(x) := \sum_{k=1}^n \chi_{E_k}(x) \leq \lim_{n \rightarrow \infty} \nu_n(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) := \nu(x).$$

The average of  $\nu_n$  over  $\Omega$  is given by

$$A_n = \int_{\Omega} \nu_n(x) dx = \sum_{k=1}^n \int_{\Omega} \chi_{E_k}(x) dx = \sum_{k=1}^n |E_k| \quad (5)$$

and

$$\int_{\Omega} \frac{\nu_n(x)}{A_n} dx = \frac{\int_{\Omega} \nu_n(x) dx}{\int_{\Omega} \nu_n(x) dx} = 1. \quad (6)$$

Thus the average of  $\nu_n(x)/A_n := f_n(x)$  over  $\Omega$  is 1. By (3) and (5),  $A_n \rightarrow \infty$  as  $n \rightarrow \infty$  and so  $\nu(x)$  can be infinite. Now the  $\limsup E$  is given by

$$E = \{x \in \Omega: \sum_{n=1}^{\infty} \chi_{E_n}(x) = \infty\} = \{x \in \Omega: \nu(x) = \infty\},$$

so that

$$\nu(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) < \infty \text{ if and only if } x \notin E,$$

i.e.,  $\nu(x) < \infty$  if and only if  $x \in \Omega \setminus E = E^c$ . But by hypothesis,  $A_n \rightarrow \infty$  as  $n \rightarrow \infty$ , whence for each  $x \in E^c$ ,  $f_n(x) = \nu(x)/A_n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus the contribution to  $\int_{\Omega} f_n = \int_E f_n + \int_{E^c} f_n = 1$  is mainly from  $E$ . Assume for the moment that, as one would expect,

$$\int_E f_n = \frac{1}{A_n} \int_E \nu_n \rightarrow 1 \text{ as } n \rightarrow \infty, \quad (7)$$

so that from Cauchy's inequality,

$$1 \leq \frac{(\int_E \nu_n(x)^2 dx)^{1/2} |E|^{1/2}}{A_n} + o(1),$$

whence on rearranging,

$$\limsup_{n \rightarrow \infty} \frac{A_n^2}{\int_E \nu_n(x)^2 dx} \geq 1.$$

But

$$\begin{aligned} \int_{\Omega} \nu(x)^2 dx &= \int_{\Omega} \sum_{k=1}^n \chi_{E_k}(x) \sum_{l=1}^n \chi_{E_l}(x) dx = \int_{\Omega} \sum_{k,l=1}^n \chi_{E_k}(x) \chi_{E_l}(x) dx \\ &= \int_{\Omega} \sum_{k,l=1}^n \chi_{E_k \cap E_l}(x) dx = \sum_{k,l=1}^n |E_k \cap E_l|, \end{aligned}$$

and the result follows.

To prove that  $\int_{E^c} f_n = o(1)$  and hence prove (7), consider the sequence  $(f_n = \nu_n/A_n : E^c \rightarrow \mathbb{R})$  of (measurable) non-negative functions. This sequence is well defined with  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for each  $x \in E^c$ . Further  $|E^c| \leq 1$ . Hence by Egoroff's theorem, given  $\eta > 0$ , there exists a measurable subset  $F_{\eta} \subseteq E^c$  such that  $|F_{\eta}| < \eta$  and  $f_n \rightarrow 0$  uniformly on  $E^c \setminus F_{\eta}$ . Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{E^c} f_n(x) dx &= \lim_{n \rightarrow \infty} \int_{E^c \setminus F_{\eta}} f_n(x) dx + \lim_{n \rightarrow \infty} \int_{F_{\eta}} f_n(x) dx \\ &= \int_{E^c \setminus F_{\eta}} \lim_{n \rightarrow \infty} f_n(x) dx + \lim_{n \rightarrow \infty} \int_{F_{\eta}} f_n(x) dx \\ &= 0 + \lim_{n \rightarrow \infty} \int_{F_{\eta}} f_n(x) dx. \end{aligned}$$

But  $\int_{\Omega} f_n = 1$  for each  $n$ , whence  $\int_{F_{\eta}} f_n \rightarrow 0$  as  $\eta \rightarrow 0$ . Hence

$$\lim_{n \rightarrow \infty} \int_{E^c} f_n(x) dx = \lim_{n \rightarrow \infty} \int_{E^c} \frac{\nu_n(x)}{a_n} dx = 0,$$

as required □

**Pairwise independence.** The sets  $E_k, E_l$  are *pairwise quasi independent* if there exists a constant  $C$  such that for all distinct  $k, l$ ,

$$|E_k \cap E_l| \leq C |E_k| |E_l|$$

and are *pairwise independent* if for all distinct  $k, l$ ,

$$|E_k \cap E_l| = |E_k| |E_l|.$$

The results below are immediate consequences of (4) and (3).

**Corollary 1.** *If the  $E_j$  are pairwise quasi independent, then  $|E| > 0$ .*

**Corollary 2.** *If the  $E_j$  are pairwise independent,  $E = 1$ .*

It can be shown that the  $B(q; \rho)$  are pairwise quasi-independent, *i.e.*, for distinct  $q, q'$ ,

$$|B(q; \rho) \cap B(q'; \rho')| \leq K |B(q; \rho)| |B(q'; \rho')|,$$

so that by Corollary 1,  $|W(\psi(q))| > 0$ . Establishing pairwise quasi independence involves some lengthy and difficult technicalities and will be omitted (proofs are given in [27, 28]). However, once established, the full result follows from an ergodic-type theorem of Gallagher [17] or from Lebesgue density that  $|W(\psi)| = 1$ . This ‘all or nothing’ or ‘0-1’ law was originally proved by Khintchine (who used continued fractions [21], limiting the proof to  $\mathbb{R}$ ). Other proofs based on pairwise quasi-independence or mean and variance arguments (see for example [6, Chaper VII]) in conjunction with density or ergodic ideas can be extended to higher dimensions and other generalisations. Recently Cornelia Drutu has used pairwise quasi-independence for Diophantine approximation in a symmetric spaces setting [12].

## 2. HIGHER DIMENSIONS

In higher dimensions ( $\mathbb{R}^n$ ), there are two natural forms of Diophantine approximation. First, in the simultaneous form, one considers the set of points  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  for which the inequality

$$\|q\mathbf{x}\| := \max\{|qx_1|, \dots, |qx_n|\} < \psi(q) \quad (8)$$

holds for infinitely many positive integers  $q$ . There is also the *dual* form in which one considers the proximity of the point  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  to the hyperplane

$$\{\mathbf{u} \in \mathbb{R}^n : \mathbf{q} \cdot \mathbf{u} = p\}, \quad (9)$$

where  $p \in \mathbb{Z}, \mathbf{q} \in \mathbb{Z}^n$ . More precisely, one considers the solubility of the inequality

$$|\mathbf{q} \cdot \mathbf{x} - p| < \psi(|\mathbf{q}|), \quad (10)$$

where for each  $\mathbf{u} \in \mathbb{R}^n$ ,  $|\mathbf{u}| := |\mathbf{u}|_\infty = \max\{|u_1|, \dots, |u_n|\}$ , for infinitely many  $p, \mathbf{q}$ . This notation should not be confused with that for the Lebesgue measure of a set. For convenience, introduce the resonant set

$$R_{\mathbf{q}}(p) := \{\mathbf{u} \in [0, 1]^n : |\mathbf{q} \cdot \mathbf{u} - p| = 0\}, \quad p \in \mathbb{Z} \quad (11)$$

and denote the collection of resonant sets associated with  $\mathbf{q}$  by

$$R_{\mathbf{q}} := \{\mathbf{u} \in [0, 1]^n : \|\mathbf{q} \cdot \mathbf{u}\| = 0\} = \bigcup_p R_{\mathbf{q}}(p) \quad (12)$$

and let  $B(\mathbf{q}; \rho)$  be the  $\rho$ -neighbourhood  $\{\mathbf{u} \in [0, 1]^n : \|\mathbf{q} \cdot \mathbf{u}\| < \rho\}$  of  $R_{\mathbf{q}}$ . The above two forms of approximation are special cases of a system of linear forms, discussed in §2.1. Analytical ideas, including Fourier series, play an important part in metrical Diophantine approximation, as do geometrical ones, particularly so in the dual form of Diophantine approximation.

**2.1. The Khintchine-Groshev Theorem.** Khintchine's theorem has a very general extension, originally proved by A. V. Groshev [27] with a stronger monotonicity condition, which includes as special cases simultaneous Diophantine approximation and its dual, as mentioned above. It treats real  $m \times n$  matrices  $X = (x_{ij})$ , regarded as points in  $\mathbb{R}^{mn}$ , which are  $\psi$ -approximable, *i.e.*, which satisfy

$$\|\mathbf{q}X\| < \psi(|\mathbf{q}|), \quad (13)$$

for infinitely many  $\mathbf{q} \in \mathbb{Z}^m$ , where  $\mathbf{q}X$  is a vector of the following linear forms

$$(q_1 x_{11} + \dots + q_m x_{m1}, \dots, q_1 x_{1n} + \dots + q_m x_{mn})$$

and  $\|\mathbf{u}\| = (|\|u_1\||, \dots, |\|u_n\||) = \max\{\|u_1\|, \dots, \|u_n\|\}$ . As the set of  $\psi$ -approximable points is translation invariant under integer vectors, we can restrict attention to the  $mn$ -dimensional torus  $\mathbb{T}^{mn}$ , *i.e.*, the  $mn$ -dimensional unit cube with opposite sides identified. The set of  $\psi$ -approximable points in  $\mathbb{T}^{mn}$  will be denoted by

$$W(\psi; m, n) = \{X \in \mathbb{T}^{mn} : \|\mathbf{q}X\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^m\}.$$

To avoid complicated notation, the dependence of  $W(\psi; m, n)$  on  $m, n$  will usually be omitted.

**Theorem 4.** *The  $mn$ -dimensional Lebesgue measure of  $W(\psi)$  is given by*

$$|W(\psi)| = \begin{cases} 0, & \text{if } \sum_{k=1}^{\infty} k^{m-1} \psi(k)^n < \infty, \\ 1, & \text{if } \sum_{k=1}^{\infty} k^{m-1} \psi(k)^n = \infty \text{ and when } m = 1, 2 \text{ } \psi(k) \text{ is decreasing.} \end{cases}$$

The proof is straightforward when the ‘volume’ sum

$$\sum_{k=1}^{\infty} k^{m-1} \psi(k)^n \quad (14)$$

converges, as the fact that  $W(\psi)$  can be expressed as a lim sup set again provides a direct and simple proof that  $W(\psi)$  has measure 0. However not surprisingly, as in one dimension, the case of divergence is much more difficult and the more general lower bound for lim sup sets is used.

**2.2. Torus geometry in the plane.** Geometrical ideas play a particularly important role in the dual form of Diophantine approximation, in which the proximity of the point  $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$  to the hyperplane

$$\{\mathbf{u} \in \mathbb{R}^m : \mathbf{q} \cdot \mathbf{u} = p\}, \quad (15)$$

where  $p \in \mathbb{Z}$ ,  $\mathbf{q} \in \mathbb{Z}^m$  is considered. More precisely, one considers the solubility of the inequality

$$|\mathbf{q} \cdot \mathbf{x} - p| < \psi(|\mathbf{q}|), \quad (16)$$

where  $|\mathbf{q}| = \max\{|q_j| : j = 1, \dots, m\}$ , for infinitely many  $\mathbf{q}$ .

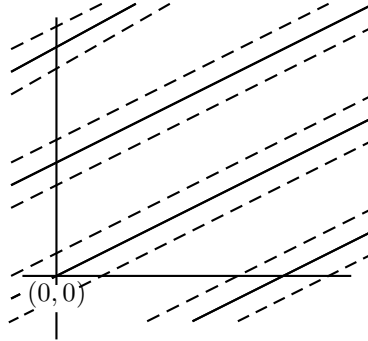


Figure 2. Resonant sets  $R_{(1,-2)}$  (bold lines) and boundaries (dashed lines) of the strips  $B_{\psi(2)}(1, -2)$  in  $\mathbb{R}^2$

It turns out that there is a precise correspondence between probabilistic independence and (algebraic) linear independence. To illustrate these ideas, the Khintchine-Groshev theorem will be considered in detail for the planar case, where they are particularly clear.

**Theorem 5.** *The Lebesgue measure of*

$$W(\psi) := \{\mathbf{u} \in [0, 1]^2 : \|\mathbf{q} \cdot \mathbf{u}\| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^2\}$$

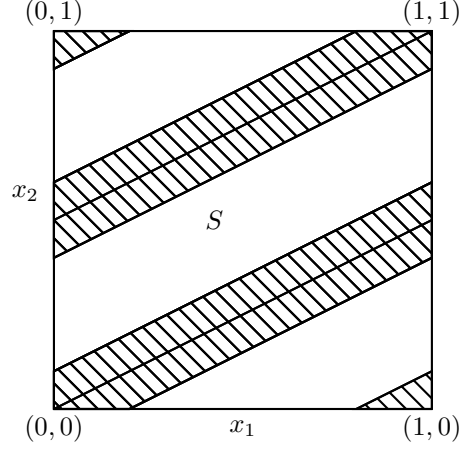
satisfies

$$|W(\psi)| = \begin{cases} 0, & \text{if } \sum_{k=1}^{\infty} k \psi(k) < \infty, \\ 1, & \text{if } \sum_{k=1}^{\infty} k \psi(k) = \infty \text{ and } \psi(k) \text{ non-increasing.} \end{cases}$$

Suppose  $\mathbf{q} \neq \mathbf{0}$ , with say  $q_1 \neq 0$ . Then the resonant set  $R(\mathbf{q})$  is a set of  $q_1$  parallel lines in  $\mathbb{T}^2$ , a distance  $1/|q_1|$  apart in the  $x_1$  direction (Figure 2). These define  $|q_1|$  strips  $S$  (from the top of a shaded strip to the top of the adjacent shaded strip) and  $\mathbb{T}^2 = \bigcup S$ . The set  $B(\mathbf{q}, \rho)$  of shaded strips  $\tilde{S}$  each of length  $\rho/q_1$  (in the  $x_1$  direction), and the ratio

$$|B(\mathbf{q}, \rho)| : |\mathbb{T}^2| = |\bigcup \tilde{S}| : |\bigcup S| = |\tilde{S}| : |S| = 2\rho : 1$$

(see Figure 3).


 Figure 3. Strips  $B_\rho(1, -2)$  in  $\mathbb{T}^2$ 

Thus

$$|B(\mathbf{q}; \rho)| = 2\rho = |(-\rho, \rho)|, \quad (17)$$

as in the case  $m = 1$ . The extension to the general case

$$|B(\mathbf{q}; \rho)| = 2^n \rho^n = |(-\rho, \rho)|^n,$$

follows by considering  $n$  copies of the 2-dimensional space spanned by  $\mathbf{q}$  and  $\mathbf{q}'$  and the volume of the corresponding  $mn$ -dimensional prisms. The determination of the Lebesgue measure of  $W(\psi)$  in the case of convergence follows readily. Fourier series can also be used (see [27]).

### 3. CONVERGENCE AND MEASURE 0

The ‘probabilistic’ interpretation discussed above, in which the Lebesgue measure  $|B(\mathbf{q}, \psi(|\mathbf{q}|))|$  of the set  $B(\mathbf{q}, \psi(|\mathbf{q}|))$  is interpreted as the probability that  $\mathbf{x} \in \mathbb{T}^n$  satisfies  $\|\mathbf{q} \cdot \mathbf{x}\| < \psi(|\mathbf{q}|)$ , reduces the result to the Borel-Cantelli Lemma. For convenience, write

$$B_{\mathbf{q}} := B(\mathbf{q}; \psi(|\mathbf{q}|)) \quad (18)$$

and consider the sets  $B_{\mathbf{q}}$ ,  $\mathbf{q} \in \mathbb{Z}^2 \setminus \{0\}$ , as a sequence  $E_r$ ,  $r = 1, 2, \dots$  in  $\mathbb{T}^2$  by ordering the integer vectors  $\mathbf{q}$ , so that  $\mathbf{q} = \mathbf{q}(r)$  is the  $r$ -th vector in  $\mathbb{Z}^m \setminus \{0\}$ . Then by (17)

$$|B_{\mathbf{q}}| = 2\psi(|\mathbf{q}|),$$

so that

$$\begin{aligned} \sum_{r=1}^{\infty} |E_r| &= \sum_{r=1}^{\infty} |B_{\mathbf{q}(r)}| = \sum_{r=1}^{\infty} 2\psi(|\mathbf{q}(r)|) = 2 \sum_{k=1}^{\infty} \sum_{|\mathbf{q}|=k} \psi(|\mathbf{q}|) = \sum_{k=1}^{\infty} \psi(k) \sum_{|\mathbf{q}|=k} 1 \\ &\asymp \sum_{k=1}^{\infty} k\psi(k), \end{aligned}$$

since there are  $2(2k+1) \asymp k^2$  non-zero integer vectors with  $|\mathbf{q}| = k$  (positive quantities  $a, b$  are comparable, denoted by  $a \asymp b$ , if there are constants  $K, K'$  such that  $a \leq Kb$  and  $b \leq K'a$ ). The convergence of the volume sum (14) thus implies the convergence of the measure sum  $\sum_{r=1}^{\infty} |E_r|$  and hence that  $|E| = |W(\psi)| = 0$ . It is clear that the proof extends to the general case.

## 4. DIVERGENCE AND FULL MEASURE

In the harder case when (14) diverges, it turns out that when  $m \geq 2$ , the pairwise ‘probabilistic’ independence of sets is associated with linearly independent pairs of integer vectors, *i.e.*, pairs of vectors which are not collinear with the origin. Thus the more general version of the divergence part of the Borel-Cantelli Lemma (Corollary 1 to Theorem 3) can be used. In this argument, the monotonicity condition can be relaxed when  $m \geq 3$ . Gallagher [17] has shown that, under a weak coprimality condition, monotonicity can be dropped when  $n \geq 2$  and  $m = 1$  (the simultaneous case). Indeed even more general results, where the argument of the error function is the vector  $\mathbf{q}$  rather than its supnorm  $|\mathbf{q}|$ , for  $m + n > 2$  were obtained for primitive solutions  $\mathbf{q}$  by Sprindžuk [27] and Schmidt [19]. Note that the Duffin-Schaeffer conjecture holds for the simultaneous case [25]. Consider another vector  $\mathbf{q}'$  and suppose that  $\mathbf{q}, \mathbf{q}' \in \mathbb{Z}^2$  are linearly independent (see Figure 4). Then the set  $B(\mathbf{q}, \rho) \cap B(\mathbf{q}', \rho')$  tessellates  $\mathbb{T}^2$  into  $|\mathbf{q} \times \mathbf{q}'|$  parallelograms  $\Pi$  say, each of area  $1/|\mathbf{q} \times \mathbf{q}'|$ . Thus

$$\mathbb{T}^2 = \cup \Pi.$$

In addition, the set  $B(\mathbf{q}; \rho) \cap B(\mathbf{q}', \rho')$  is the union  $\cup \tilde{\Pi}$  of  $|\mathbf{q} \times \mathbf{q}'|$  parallelograms  $\tilde{\Pi}$  (shown doubly hatched in Figure 4) each of area  $\rho\rho'/|\mathbf{q} \times \mathbf{q}'|$ . By similarity, the ratio

$$|\cup \tilde{\Pi}| : |\cup \Pi| = |\tilde{\Pi}| : |\Pi| = 4\rho\rho' : 1.$$

But  $|\cup \Pi| = 1$  and  $|\cup \tilde{\Pi}| = |B(\mathbf{q}; \rho) \cap B(\mathbf{q}', \rho')|$ , whence

$$|B(\mathbf{q}; \rho) \cap B(\mathbf{q}', \rho')| = 4\rho\rho' = |B(\mathbf{q}; \rho)| |B(\mathbf{q}'; \rho')| \quad (19)$$

and  $B(\mathbf{q}; \rho), B(\mathbf{q}', \rho')$  are independent (see figure 4).

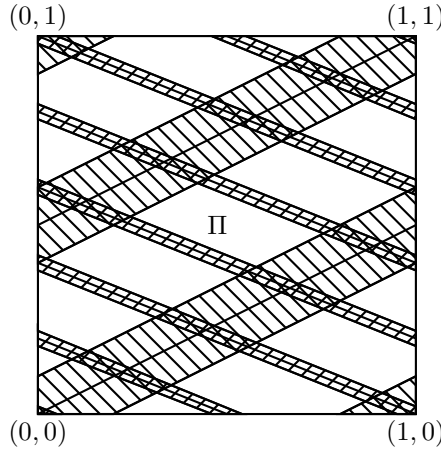


Figure 4.  $B(1, -2; \rho) \cap B(2, 5; \rho')$  in  $\mathbb{T}^2$

The extension to the general case again follows by considering  $n$  copies of the 2-dimensional space spanned by  $\mathbf{q}$  and  $\mathbf{q}'$  and the volume of the corresponding  $mn$ -dimensional prisms. This gives for  $m \geq 2$ ,

$$|B(\mathbf{q}; \rho) \cap B(\mathbf{q}'; \rho')| = |B(\mathbf{q}; \rho)| \cdot |B(\mathbf{q}'; \rho')| = 2^{2n} \rho^n \rho'^n.$$

Thus when  $m \geq 2$  the pairwise probabilistically independent vectors  $\mathbf{q}$  in  $\mathbb{Z}^n$  are precisely the pairwise linearly independent integer vectors in  $\mathbb{Z}^n$ . Thus to apply Theorem 3, we need a ‘large’ set of such vectors and to find one, some number theoretic ideas are required.

**4.1. A set of pairwise linearly independent vectors.** The following argument is drawn from [27]. Let the highest common factor of the integer components  $q_1, \dots, q_m$  of  $\mathbf{q} \in \mathbb{Z}^m$  be denoted by  $(\mathbf{q})$ , so that  $(\mathbf{q}) = \text{hcf}(q_1, \dots, q_m)$ . The vector  $\mathbf{q} \in \mathbb{Z}^m$  is said to be *primitive* if  $(\mathbf{q}) = \pm 1$ . Two distinct primitive vectors,  $\mathbf{q}, \mathbf{q}'$  say, are linearly independent when  $\mathbf{q}, \mathbf{q}'$  are primitive. For if  $\mathbf{q}, \mathbf{q}'$  are linearly dependent, then  $a\mathbf{q} = a'\mathbf{q}'$  for some real  $a, a'$ ,  $a, a'$  can be assumed to be coprime integers (*i.e.*, integers with no common factors other than  $\pm 1$ ). Thus  $a'$  divides each component  $q_1, \dots, q_m$  of  $\mathbf{q}$  and  $a$  divides each component  $q'_1, \dots, q'_m$  of  $\mathbf{q}'$ . Since  $\mathbf{q}, \mathbf{q}'$  are primitive,  $a, a' = \pm 1$ .



If in addition,  $q_m, q'_m \geq 1$ , then  $a = a' = 1$  and  $\mathbf{q} = \mathbf{q}'$ . In other words, no pair of distinct integer vectors in the set

$$\mathcal{P}_N = \{\mathbf{q} \in \mathbb{Z}^m : \mathbf{q} \text{ primitive}, |\mathbf{q}| \leq N, q_m \geq 1\} = \{\mathbf{q} \in \mathbb{Z}^m : (\mathbf{q}) = 1, |\mathbf{q}| \leq N, q_m \geq 1\}$$

is linearly dependent. This set is the union of disjoint subsets (or 'hemispheres')  $S_k$ , consisting of vectors  $\mathbf{q}$  in  $\mathcal{P}_N$  with 'radius'  $|\mathbf{q}| = k$ , i.e.,

$$\mathcal{P}_N = \bigcup_{k=1}^N S_k.$$

Now let

$$\mathcal{P}_\infty = \{\mathbf{q} \in \mathbb{Z}^m : (\mathbf{q}) = 1, q_m \geq 1\} = \bigcup_{k=1}^{\infty} S_k.$$

Then distinct vectors  $\mathbf{q}, \mathbf{q}' \in \mathcal{P}_\infty$  are linearly independent and so  $B_{\mathbf{q}}, B_{\mathbf{q}'}$  are independent, i.e.,  $|B_{\mathbf{q}} \cap B_{\mathbf{q}'}| = |B_{\mathbf{q}}||B_{\mathbf{q}'}|$ .

The number of vectors  $\mathbf{q}$  in  $\mathbb{Z}^m$  with  $|\mathbf{q}| = k$  and  $q_m \geq 1$  is  $2(m-1)(2k+1)^{m-2}k$ , since each coordinate  $q_j$ ,  $j = 1, \dots, m$ , satisfies  $|q_j| \leq k$  and  $|q_{j'}| = k$  for some  $j'$ ,  $1 \leq j' \leq k$ . To obtain an asymptotic formula for  $\text{Card } S_k$ , divide up these vectors  $\mathbf{q}$  in  $\mathbb{Z}^m$  into classes  $S(h)$  where  $h|k$  ( $h$  divides  $k$ ). Then a vector  $\mathbf{q} \in S(h)$  is of the form

$$\mathbf{q} = (q_1, \dots, q_m) = (hr_1, \dots, hr_m) = h\mathbf{r},$$

where  $hr_{j'} = q_{j'} = k$  and  $\mathbf{r}$  is primitive ( $(\mathbf{r}) = 1$ ). Now

$$\begin{aligned} \text{Card } S_k &= \sum_{\mathbf{q} \in \mathcal{P}_N, |\mathbf{q}|=k} 1 = \sum_{(\mathbf{q})=1, |\mathbf{q}|=k, q_m \geq 1} 1 = \sum_{\substack{\mathbf{q} \in \mathbb{Z}^m \\ |\mathbf{q}|=k, q_m \geq 1}} \sum_{d|(\mathbf{q})} \mu(d) \\ &= \sum_{d|k} \mu(d) \sum_{\substack{|\mathbf{r}|=k/d \in \mathcal{P}_N \\ r_m \geq 1}} 1 = \sum_{d|k} \mu(d) \left(2\frac{k}{d} + 1\right)^{m-2} \frac{k}{d} 2(m-1) \\ &= 2^{m-1}(m-1) \sum_{d|k} \mu(d) (k/d)^{m-1} + O\left(k^{m-2} \sum_{d|k} \frac{|\mu(d)|}{d^{m-2}}\right), \end{aligned}$$

where  $\mu(d)$  is the Möbius function [18, p 234], given by

$$\mu(d) = \begin{cases} (-1)^r & \text{when } d \text{ is the product of } r \text{ distinct primes,} \\ 0 & \text{otherwise,} \end{cases}$$

and has the important property that  $\sum_{d|k} \mu(d) = 1$  when  $k = 1$  and 0 otherwise. But  $\varphi(k)$ , the number of integers less than  $k$  and coprime to  $k$ , is given by

$$\varphi(k) := \sum_{1 \leq j \leq k, (j,k)=1} 1 = k \sum_{d|k} \frac{\mu(d)}{d},$$

whence for  $m = 2$ ,

$$\text{Card } S_k = 2k \sum_{d|k} \mu(d) (1/d) + O\left(\sum_{d|k} |\mu(d)|\right) = 2\varphi(k) + O(d(k)) \asymp \varphi(k),$$

since  $d(k) = \sum_{d|k} 1$ , the number of divisors of  $k$ , satisfies  $d(k) = O(k^\delta)$  for any positive  $\delta$  [18, Theorem 315]. When the real part of the complex number  $z > 1$ , Riemann's zeta function is given by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^z}\right)^{-1},$$

so that for  $m \geq 3$ ,

$$\frac{1}{\zeta(m-1)} = \left( \sum_{k=1}^{\infty} \frac{1}{k^{m-1}} \right)^{-1} < \prod_{\substack{p|k \\ p \text{ prime}}} \left( 1 - \frac{1}{p^{m-1}} \right) = \sum_{d|k} \frac{\mu(d)}{d^{m-1}} < 1.$$

Thus

$$\text{Card } S_k \asymp \begin{cases} \varphi(k), & m = 2 \\ k^{m-1}, & m \geq 3. \end{cases}$$

Now, as is well known,  $\varphi(k)$  is comparable ‘on average’ to  $k$  [18, Theorem 330] or to be precise,

$$\Phi(N) = \sum_{k=1}^N \varphi(k) = \frac{3}{\pi^2} N^2 + O(N \log N) \asymp N^2 \quad (20)$$

(see [18, 27]) and it turns out that if  $\psi$  is non-increasing, the volume sum determines the Lebesgue measure of  $W(\psi)$ .

As has been said, Fourier analysis of the periodic function  $\chi_{B(\mathbf{q}; \rho)}$  can also be used; the linear independence of the  $\mathbf{q}$ ’s is crucial in establishing the measure of the intersection  $B(\mathbf{q}; \rho) \cap B(\mathbf{q}'; \rho')$  (see [27, Chapter 1, §5] for details).

**4.2. Completing the proof of Khintchine-Groshev theorem.** Next the divergence of the volume sum (14) is shown to imply the divergence of a related sum over independent integer vectors. For each  $N = 1, 2, \dots$ , the partial sum

$$\begin{aligned} \sum_{\mathbf{q} \in \mathcal{P}_N} |B_{\mathbf{q}}| &= 2^n \sum_{\mathbf{q} \in \mathcal{P}_N} \psi(|\mathbf{q}|)^n = 2^n \sum_{k=1}^N \sum_{\mathbf{q} \in S_k} \psi(|\mathbf{q}|)^n = 2^n \sum_{k=1}^N \psi(k)^n \sum_{\mathbf{q} \in S_k} 1 = 2^n \sum_{k=1}^N \psi(k)^n \text{Card } S_k \\ &\asymp \begin{cases} \sum_{k=1}^N \varphi(k) \psi(k)^n & \text{when } m = 2 \\ \sum_{k=1}^N k^{m-1} \psi(k)^n & \text{when } m \geq 3. \end{cases} \end{aligned}$$

Thus when  $m \geq 3$ , the divergence of the sum (14) implies the divergence of  $\sum_{\mathbf{q} \in \mathcal{P}_{\infty}} |B_{\mathbf{q}}|$ . To deal with the case  $m = 2$ , we use  $\Phi(N) \asymp N^2$  [18] and that the monotonicity of  $\psi(k)$  implies that  $\sum_{k=1}^N \varphi(k) \psi(k)^n$  is comparable to  $\sum_{k=1}^N k \psi(k)^n$ . Hence if  $\psi(k)$  is decreasing, then

$$\sum_{\mathbf{q} \in \mathcal{P}_N} |B_{\mathbf{q}}| \asymp \sum_{k=1}^N \varphi(k) \psi(k)^n \asymp \sum_{k=1}^N k \psi(k)^n$$

and the divergence of the right hand sum implies the divergence of the left hand sum, which in turn implies  $|W(\psi)| = 1$ .

There are two interesting refinements of the Khintchine-Groshev theorem. The first arises in the divergent case and is a quantitative version in the sense of an asymptotic formula for the number of solutions [19, 27]. In the case of the real numbers, the number  $\mathcal{N}(N; \alpha)$  of solutions with  $q \leq N$  of the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q},$$

where again  $\psi$  is decreasing, is

$$\mathcal{N}(N; \alpha) = 2 \sum_{q=1}^N \psi(q) (1 + o(1)).$$

For simultaneous Diophantine approximation, where monotonicity can be omitted for dimensions at least 2, an asymptotic formula holds for dimension at least 3 [17]. Asymptotic formulae will not be discussed here but details are in [19, 27]. The other refinement concerns the finer structure of the null set when the series converges.

## 5. HAUSDORFF DIMENSION

It is a familiar fact that the one-dimensional Lebesgue measure of a unit segment is 1 but that the two dimensional or planar Lebesgue measure is 0. This simple example illustrates that a dimension is associated with the determination of Lebesgue measure. For standard shapes, such as the real line, the plane, rectangle or circle, the dimension is the topological dimension and so is integral and obvious (from our point of view, it could be called the Lebesgue dimension). Of course, it is the Lebesgue measure of a set which is of interest. As is well known, exceptional sets of  $n$ -dimensional Lebesgue measure zero can be studied using the more delicate notions of Hausdorff dimension and measure, which allow null sets to be distinguished. Hausdorff dimension, which is defined in terms of Hausdorff measure, is a generalisation of Lebesgue dimension and the two notions coincide for standard sets. However, they differ in that any set in finite dimensional Euclidean space has a Hausdorff dimension (which in general will not be an integer). In particular null sets have a Hausdorff dimension, thus offering a way of studying sets that are 'invisible' or 'negligible' in terms of Lebesgue measure and of distinguishing between them. By contrast, a set of positive Lebesgue measure has full Hausdorff dimension (equal to the Lebesgue dimension of the ambient space).

Although conceptually a simple but profound extension of Carathéodory's construction of Lebesgue measure, Hausdorff measure has a somewhat complicated definition and the reader is referred to [5, 14, 15, 16, 24, 26] for fuller accounts. For completeness a simpler formulation suited to our purposes will be sketched. Let  $\mathcal{C}$  be a finite or countable collection of open hypercubes  $C \subset \mathbb{R}^k$  with sides of length  $\ell(C)$  and parallel to the axes. For each non-negative real number  $s$  the  $s$ -volume of the collection  $\mathcal{C}$  is defined to be

$$\ell^s(\mathcal{C}) = \sum_{C \in \mathcal{C}} \ell(C)^s.$$

For any set  $E$  in  $\mathbb{R}^k$  and any real number  $\delta > 0$ , let  $\mathcal{H}_\delta^s(E) = \inf \ell^s(\mathcal{C}_\delta)$  be the infimum taken over all 'approximating' covers  $\mathcal{C}_\delta$  of  $E$  by hypercubes  $C$  with side length  $\ell(C)$  at most  $\delta$ . When  $0 < \delta < 1$  and  $t < s$ ,  $\ell(C)^s < \ell(C)^t$  and so the number  $\mathcal{H}_\delta^s(E)$  decreases as  $s$  increases. The  $s$ -dimensional outer measure  $\mathcal{H}^s(E)$  of  $E$  defined by

$$\mathcal{H}^s(E) = \sup\{\mathcal{H}_\delta^s(E) : \delta > 0\} = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E)$$

is comparable to Hausdorff outer measure. If  $t > s$ , then  $\ell(C)^t \leq \delta^{t-s} \ell(C)^s$ , whence  $\mathcal{H}_\delta^t(E) \leq \delta^{t-s} \mathcal{H}_\delta^s(E)$ , so that when  $\mathcal{H}^t(E)$  is positive,  $\mathcal{H}^s(E)$  is infinite and when  $\mathcal{H}^s(E)$  is finite,  $\mathcal{H}^t(E)$  vanishes. The Hausdorff dimension  $\dim E$  of  $E$  is defined by

$$\dim E = \inf\{s \in \mathbb{R} : \mathcal{H}^s(E) = 0\},$$

so that

$$\mathcal{H}^s(E) = \begin{cases} \infty, & s < \dim E, \\ 0 & s > \dim E. \end{cases}$$

Thus the dimension is that value of  $s$  at which  $\mathcal{H}^s(E)$  'drops' discontinuously from infinity (see figure 5). Determining the Hausdorff measure at this value is not always easy and will not be discussed (but see §6 below). A cover for  $E$  serves as a cover for any subset  $E'$  of  $E$  and so  $E' \subseteq E$  implies that

$$\dim E' \leq \dim E.$$

A comparison with viewing an object under a microscope can be made. If the microscope lens is too close to the object  $E$ , the image fills the eyepiece and cannot be resolved; if the lens is too far away, the image is invisible. At the focal length (*i.e.*, at  $s = \dim E$ ), the image is in focus and can be seen properly. Thus the Hausdorff dimension is like the focal length – a Hausdorff measure can be assigned to the set under consideration at the Hausdorff dimension (see figure 5).

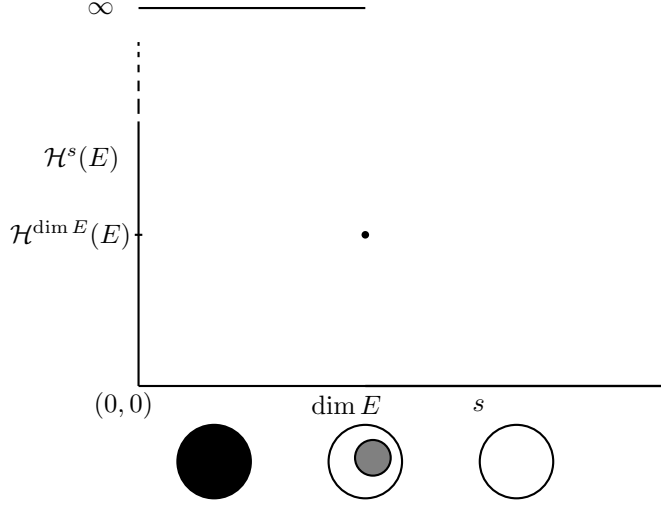


Figure 5. Graph of Hausdorff measure  $\mathcal{H}^s(E)$  against exponent  $s$ . The three circles represent views through a microscope.

When  $s$  is a non-negative integer  $m$  say, Hausdorff's  $m$ -measure is comparable with Lebesgue's  $m$ -dimensional measure (and they agree when  $m = 1$ ). The Hausdorff dimension of a set  $A \subseteq \mathbb{R}^k$  of Lebesgue measure 0 is often established by obtaining an upper and a lower inequality which combine to give the desired equality. In the case of lim sup sets, such as  $W(\psi)$ , the upper bound usually follows straightforwardly from a natural cover arising from the definition and is closely related to the cover used in determining the Lebesgue measure. For simplicity, let us take  $\psi(r) = r^{-v}$ , where  $v > 0$ , and write  $W(\psi) = W_v$ . Then for  $W_v$ , the cover  $\mathcal{C}$  of hypercubes arises from the sets  $B_{\mathbf{q}}$ , where  $\mathbf{q}$  is a non-zero integer vector and a straightforward calculation of the  $s$ -volume  $\ell^s(\mathcal{C})$  of the cover (see for example [9]) gives

$$\dim W_v \leq \begin{cases} (m-1)n + \frac{m+n}{v+1} & \text{when } v > \frac{m}{n} \\ mn & \text{when } v \leq \frac{m}{n}. \end{cases} \quad (21)$$

The lower bound is usually more difficult. In the case of  $W_v$ , the argument can be shortened considerably by using the idea of 'ubiquity'. This was introduced originally to systematise and extend the determination of the lower bound for the Hausdorff dimension of sets of number theoretic and physical interest [11].

**5.1. Ubiquity.** We start with some definitions and then introduce a lim sup set which is associated with  $W(\psi)$  and easier to work with. Denote the  $\delta$ -neighbourhood of the resonant set  $R_{\mathbf{q}}$  by

$$\tilde{B}(\mathbf{q}; \delta) = \{X \in \mathbb{T}^{mn} : \text{dist}_{\infty}(X, R_{\mathbf{q}}) < \delta\}$$

where  $\text{dist}_{\infty}(X, R_{\mathbf{q}}) = \inf\{\text{dist}_{\infty}(X, U) : U \in R_{\mathbf{q}}\}$  is the distance in the supremum norm from  $X$  to  $R_{\mathbf{q}}$ . This is not the same set as the set

$$B(\mathbf{q}; \delta) := \{X \in \mathbb{T}^{mn} : \|\mathbf{q}X\| < \delta\},$$

which when  $m = 1$  reduces to the set  $B(\mathbf{q}; \delta)$  defined by  $\{\mathbf{u} \in [0, 1]^m : \|\mathbf{q} \cdot \mathbf{u}\| < \delta\}$ . However it is readily shown that the sets are related by the following inclusions: when  $\mathbf{q} \neq \mathbf{0}$ ,

$$\tilde{B}(\mathbf{q}; \frac{\delta}{m|\mathbf{q}|}) \subseteq B(\mathbf{q}; \delta) \subseteq \tilde{B}(\mathbf{q}; \frac{\delta}{|\mathbf{q}|}) \quad (22)$$

Let  $\tilde{\rho} : \mathbb{N} \rightarrow (0, \infty)$  be a decreasing function. When for the family  $\mathcal{R} = \{R_{\mathbf{q}} : \mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\}$ ,

$$|\mathbb{T}^{mn} \setminus \bigcup_{1 \leq |\mathbf{q}| \leq N} \tilde{B}(\mathbf{q}; \tilde{\rho}(N))| \rightarrow 0 \text{ as } N \rightarrow \infty,$$

we say that  $\mathcal{R}$  is *ubiquitous with respect to the function  $\tilde{\rho}$* .

Let

$$\tilde{\rho}(N) = 2N^{-1-\frac{m}{n}} \log N.$$

When  $m \geq 2$ , the independence of the sets  $B(\mathbf{q}; \rho)$  implies that  $\mathcal{R}$  is ubiquitous with respect to  $\tilde{\rho}$  [10] and a general form of Dirichlet's theorem implies ubiquity without restriction on the dimension  $m$  [9, 11] (see [8] for Hausdorff measure results). In essence this means that 'most'  $X$  are within  $\tilde{\rho}(N) = 2N^{-1-m/n} \log N$  of some resonant set  $R_{\mathbf{q}}$  with  $1 \leq |\mathbf{q}| \leq N$ .

Consider the lim sup set

$$\Lambda(\tilde{\psi}) = \left\{ X \in \mathbb{T}^{mn} : |X - R_{\mathbf{q}}| < \tilde{\psi}(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^m \right\} = \bigcap_{N=1}^{\infty} \bigcup_{|\mathbf{q}|=N}^{\infty} \tilde{B}(\mathbf{q}; \tilde{\psi}(N)),$$

where  $\tilde{\psi}(N) = N^{-v-1}/m$ . By [9, 11], the ubiquity of the family  $\mathcal{R}$  with respect to  $\tilde{\rho}$  implies that the Hausdorff dimension of  $\Lambda(\tilde{\psi})$  satisfies

$$\dim \Lambda(\tilde{\psi}) \geq \dim \mathcal{R} + \gamma \operatorname{codim} \mathcal{R},$$

where  $\dim \mathcal{R}$  is the topological dimension  $(m-1)n$  of the resonant set  $R_{\mathbf{q}}$  and codimension  $\operatorname{codim} \mathcal{R} = n$  and

$$\begin{aligned} \gamma &= \min \left\{ 1, \limsup_{N \rightarrow \infty} \frac{\log \tilde{\rho}(N)}{\log \tilde{\psi}(N)} \right\} \\ &= \min \left\{ 1, \left(1 + \frac{m}{n}\right) \frac{1}{\liminf_{N \rightarrow \infty} \frac{\log(mN^{1+v})}{\log N}} \right\} \\ &= \min \left\{ 1, \frac{1 + \frac{m}{n}}{1 + v} \right\}. \end{aligned}$$

Thus

$$\dim \Lambda(\tilde{\psi}) \geq \min \left\{ mn, (m-1)n + \frac{m+n}{v+1} \right\}.$$

By (22) and the choice of  $\tilde{\psi}$ ,

$$\tilde{B}(\mathbf{q}; \tilde{\psi}(N)) = \tilde{B}(\mathbf{q}; N^{-v-1}/m) \subseteq B(\mathbf{q}; N^{-v}),$$

whence

$$\Lambda(\tilde{\psi}) \subset W_v.$$

Combining this with (21) yields

$$\dim W_v = \begin{cases} (m-1)n + \frac{m+n}{v+1} & \text{when } v > \frac{m}{n} \\ mn & \text{when } v \leq \frac{m}{n}. \end{cases}$$

In one dimension, ubiquity is essentially equivalent to the 'regular systems' introduced by Baker and Schmidt [1] and the above result reduces to the Jarník-Besicovitch theorem.

## 6. FURTHER DEVELOPMENTS

Determining the Hausdorff dimension of a set can be difficult enough and finding the Hausdorff measure can be even harder without special arguments available (such as when the Hausdorff measure coincides with Lebesgue measure). In another of his pioneering papers [20], Jarník established the Hausdorff measure analogue of Khintchine's theorem for simultaneous Diophantine approximation and showed that the Hausdorff  $s$ -measure at the critical exponent (where  $s = \dim W(\psi)$ ) is infinite. Dickinson and Velani extended this result to systems of linear forms in [8]. More recently with Beresnevich, they have developed a powerful and unifying framework for obtaining the Hausdorff measure of lim sup sets in the general setting of a compact metric space endowed with a non-atomic probability measure and containing a family of resonant sets [2]. The lim sup sets consist of points which lie close to infinitely many resonant sets and include a very wide range of results in the theory of metric Diophantine approximation, including the set  $W(\psi)$  discussed above. For recent applications, see the paper by Drutu [12] and the paper of Beresnevich

and Velani [3] in this proceedings. A similarity between the two main theorems in [2] suggests an equivalence between certain Lebesgue and Hausdorff measure results and a Hausdorff measure analogue of the Duffin-Schaeffer conjecture [13, 25]. This is treated in a subsequent paper by Beresnevich and Velani [4].

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